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Modified Large Number Hypothesis

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In the Brans–Dicke theory, a certain large number hypothesis is equivalent implicitly to an equation of state. The equation of state corresponding to Dirac's large number hypothesis, however, is not reasonable. The Whitrow–Randall relation is regarded as a modification of Dirac's large number hypothesis, but it is not in fact in keeping with Dirac's original intention to relate only a single cosmological parameter to the gravitational "constant." In view of those facts, an alternative modification of Dirac's large number hypothesis is proposed.

1. INTRODUCTION

It is well known that one of the formulations of Dirac's large number hypothesis (LNH) can be given by (Dirac, 1938)

$$G \propto H \tag{1}$$

where G stands for the gravitational "constant" and H is the Hubble parameter $\dot{R}/R$. Besides the consideration of embodying Mach's principle, the LNH aims at explaining why the ratio of the gravitational to the electric force between the electron and the proton,

$$Gm_p m_e / e^2 = 4.4 \times 10^{-40} \tag{2}$$

is so small (Weinberg, 1972). Obviously, so long as relation (1) holds, G varies over cosmic time scales and the reason that (2) is so small is simply that the universe is old.

The starting point of Dirac's LNH is based on the following observational fact:

$$\left(\frac{\hbar^2 H_0}{Gc}\right)^{1/3} \approx m_\pi \tag{3}$$

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where $\hbar$ is the Planck constant, c is the velocity of light, m_π is the pion mass, and $H_0 = H(t_0)$ stands for the present Hubble parameter. If a relation like (3) remained valid when H were used in place of H_0, one would obtain (1). The LNH (1), plus the relation

$$G n_0 m_p = H^2 \tag{4}$$

(n_0 is the present cosmic baryon number density), leads to Dirac's cosmological model:

$$R \propto t^{1/3}, \qquad G \propto t^{-1} \tag{5}$$

This result is inconsistent with observations in at least two points. First, the present rate of decrease of G given by (5) is

$$(\dot{G}/G)_0 = -3H_0 \approx -3 \times 10^{-10}/\text{year} \tag{6}$$

while the observational upper limit of the absolute value of $(\dot{G}/G)_0$ is 10^{-11}/year (Hellings, 1983). Second, (5) does not satisfy the anthropic principle, because, "if G has decreased as $1/t$, then the temperature of the earth's surface 10^9 years ago would have been above the boiling point of water," which "could have prevented the evolution of life forms capable of curiosity about the universe" (Weinberg, 1972).

There are other observational facts, for example,

$$G\rho_0 \propto 1/t_0^2 \tag{7}$$

where ρ_0 and t_0 are the present energy density and age of the universe, respectively. A generalization of (7)

$$G\rho \propto 1/t^2 \tag{8}$$

which is often called the Whitrow–Randall relation (Whitrow and Randall, 1951), is regarded as a modification of Dirac's LNH. Some investigations of it have been made (Berman and Som, 1990; Berman, 1992a,b; Beesham, 1994). It seems to me that there is an essential difference between the Whitrow–Randall relation (8) and Dirac's LNH (1). The relation (1) relates only a single cosmological parameter H to G. From it one can conclude that so long as H varies with time, so must G. That is just where the significance of the LNH lies. On the other hand, the relation (8) relates two cosmological parameters ρ and t to G, from which one cannot obtain the conclusion that G must vary with time. In fact, (8) is also required by the Friedmann models.

Another well-known fact is that the gravitational field equations, together with a given equation of state, constitute the complete equations of dynamical cosmology in both general relativity and the Brans–Dicke theory (Brans and Dicke, 1961). If we drop the restriction (4) and use a certain LNH instead of the equation of state, the fundamental equations of dynamical cosmology in the Brans–Dicke theory are still complete. Hence, to adopt a certain LNH is in fact equivalent to giving implicitly an equation of state. The present paper will show that the equation of state corresponding to Dirac's LNH is unacceptable physically. We propose an alternative modification of Dirac's LNH without deviating from his spirit, from which an acceptable equation of state and reasonable solutions can be obtained.

2. EQUATION OF STATE DERIVED FROM DIRAC'S LNH

Consider the Robertson–Walker model with zero scalar curvature and perfect fluid source. The gravitational field equations in the Brans–Dicke theory read

$$3\frac{\ddot{R}}{R} = -\frac{8\pi}{(3+2\omega)\phi}[(2+\omega)\rho + 3(1+\omega)p] - \omega\frac{\dot{\phi}^2}{\phi^2} - \frac{\ddot{\phi}}{\phi} \tag{9}$$

$$\frac{\ddot{R}}{R} + 2\frac{\dot{R}^2}{R^2} = \frac{8\pi}{(3+2\omega)\phi}[(1+\omega)\rho - \omega p] - \frac{\dot{\phi}}{\phi}\frac{\dot{R}}{R} \tag{10}$$

$$\ddot{\phi} + 3\dot{\phi}\frac{\dot{R}}{R} = \frac{8\pi}{3+2\omega}(\rho - 3p) \tag{11}$$

If Dirac's LNH held, we should have

$$\phi = \frac{\alpha}{H} \tag{12}$$

where α is a constant, and is equal to $\phi_0 H_0$. Noticing that $\dot{R}/R = H$, $\ddot{R}/R = -qH^2$, $\dot{H} = -(q+1)H^2$, $\ddot{H} = -\dot{q}H^2 + 2(q+1)^2H^3$, $\dot{\phi}/\phi = (q+1)H$, and $\ddot{\phi}/\phi = \dot{q}H$, we can write equations (9)–(11) as follows:

$$\frac{8\pi}{(3+2\omega)\alpha}[(2+\omega)\rho + 3(1+\omega)p] = -\dot{q} - [\omega(q+1)^2 - 3q]H \tag{13}$$

$$\frac{8\pi}{(3+2\omega)\alpha}[(1+\omega)\rho - \omega p] = 3H \tag{14}$$

$$\frac{8\pi}{(3+2\omega)\alpha}(\rho - 3p) = \dot{q} + 3(q+1)H \tag{15}$$

From (13)–(15) we can obtain

$$p = \frac{1}{\omega}\left[1 + \omega + \frac{6(3+2\omega)}{\omega(q+1)^2 - 6(q+2)}\right]\rho \tag{16}$$

$$\rho = \frac{\alpha}{16\pi}[-\omega(q+1)^2 + 6(q+2)]H \tag{17}$$

Equation (16) can be regarded as an equation of state. However, it is not a reasonable one. In order to see that, we only need to substitute the present values of all quantities into (16) and (17). If we conservatively take $q_0 =$ 0.5 and $\omega = 500$ (Reasenberg *et al.*, 1979), we have

$$p_0 \approx \rho_0 \approx -22.1 \frac{H_0^2}{G_0} \tag{18}$$

Here we have used the relation

$$G = \frac{2\omega + 4}{2\omega + 3} \phi^{-1} \tag{19}$$

When we take $\omega = 9$ (Liddle *et al.*, 1992), we obtain

$$p_0 \approx \frac{34}{9} \rho_0 \approx -0.4 \frac{H_0^2}{G_0} \tag{20}$$

Both the results (18) and (20) are obviously unreasonable.

3. MODIFIED DIRAC LNH

A natural generalization of Dirac's LNH can be taken as

$$\phi = \overline{\alpha} H^{-\delta} \tag{21}$$

where δ is a constant. We have now that

$$\frac{\dot{\phi}}{\phi} = \delta(q + 1)H, \qquad \frac{\ddot{\phi}}{\phi} = \delta[\dot{q}H + (\delta - 1)(q + 1)^2 H^2] \tag{22}$$

Thus equations (9)–(11) become

$$\frac{8\pi H^{\delta-2}}{(3 + 2\omega)\overline{\alpha}} [(2 + \omega)\rho + 3(1 + \omega)p] = -\delta\dot{q} - [\delta(\omega\delta + \delta - 1)$$
$$(q + 1)^2 - 3q]H \tag{23}$$

$$\frac{8\pi H^{\delta-2}}{(3 + 2\omega)\overline{\alpha}} [(1 + \omega)\rho - \omega p] = [3 + (\delta - 1)(q + 1)]H \tag{24}$$

$$\frac{8\pi H^{\delta-2}}{(3 + 2\omega)\overline{\alpha}} (\rho - 3p) = \delta\dot{q} + \delta[(\delta - 1)(q + 1) + 3]$$
$$(q + 1)H \tag{25}$$

From (23)–(25) we obtain that

$$p = \frac{1}{\omega}\left(\omega + 1 + \frac{2[3 + (\delta - 1)(q + 1)](3 + 2\omega)}{\omega\delta^2(q + 1)^2 - 6\delta(q + 1) - 6}\right)\rho \tag{26}$$

$$\rho = \frac{\overline{\alpha} H^{2-\delta}}{16\pi}[-\omega\delta^2(q + 1)^2 + 6\delta(q + 1) + 6] \tag{27}$$

Eliminating ρ and p from (23)–(25) yields

$$2\omega\delta\dot{q} = [\omega\delta(2 - \delta)(q + 1)^2 - 6(\omega\delta + 1)(q + 1) + 12]H \tag{28}$$

or, in more convenient form,

$$\frac{dH}{H} = -\frac{2\omega\delta(q + 1)d(q + 1)}{\omega\delta(2 - \delta)(q + 1)^2 - 6(\omega\delta + 1)(q + 1) + 12} \tag{29}$$

The general solution of (29) is

$$CH = \left(\frac{[q + 1 - (A - B)/\omega\delta(2 - \delta)]^{A-B}}{[q + 1 - (A + B)/\omega\delta(2 - \delta)]^{A+B}}\right)^{1/(2-\delta)B} \tag{30}$$

where C is an integral constant, and A and B are defined by

$$A = 3(\omega\delta + 1), \qquad B = [A^2 - 12\omega\delta(2 - \delta)]^{1/2} \tag{31}$$

The observational fact that $p_0 = 0$ in combination with (26) gives

$$\omega(\omega + 1)\delta^2(q_0 + 1)^2 - 2[\omega\delta + (2\omega + 3)](q_0 + 1) + 6(\omega + 2) = 0 \tag{32}$$

Considering that many other cosmological models give $\dot{q} = 0$ and that there is no distinct inconsistency between this result and the present observation, we can take $\dot{q}_0 = 0$ for the moment. Thus (28) gives

$$\omega\delta(2 - \delta)(q_0 + 1)^2 - 6(\omega\delta + 1)(q_0 + 1) + 12 = 0 \tag{33}$$

The set of equations (32) and (33) has the solutions

$$\delta = \frac{2}{3\omega + 4}, \qquad q_0 = \frac{\omega + 2}{2(\omega + 1)} \tag{34}$$

Substituting (34) into (30) gives $C = 0$, from which we immediately obtain $\dot{q} = \dot{q}_0 = 0$. The cosmological solutions are then given by

$$p = 0 \tag{35}$$

$$\rho \propto t^{-6(1+\omega)/(4+3\omega)} \tag{36}$$

$$R \propto t^{2(1+\omega)/(4+3\omega)} \tag{37}$$

$$\phi \propto t^{2/(4+3\omega)} \tag{38}$$

The solutions (36)–(38) are just those given by Brans and Dicke (1961) using the equation of state $p = 0$.

4. DISCUSSION

From (21) and (34) we can rewrite the modified Dirac LNH in the form

$$G \propto H^{2/(4+3\omega)} \tag{39}$$

It relates a single cosmological parameter H to G. In this sense we can say that it embodies Dirac's spirit.

As mentioned above, the starting point of Dirac's LNH is the observational result (3). For our modified LNH have we any relation like (3) which is used as a starting point? We think that such a relation exists and can be taken to have the form

$$\left[\frac{Gm_p m_e}{e^2}\left(\frac{G\hbar H_0^2}{c^5}\right)^{-1/(5+3\omega)}\right]^{1/(1+4\omega)} \alpha^{-4/\omega} \approx 1 \tag{40}$$

where $\alpha = e^2/\hbar c$ is the fine structure constant.

Moreover, if we believe that $\omega = 9$, we should have $q_0 = 11/20$, $G \propto t^{-2/31}$, *and* $(\dot{G}/G)_0 = -31H_0/320$. Those results do not contradict the astrophysical data within the present observational accuracy.

REFERENCES

Beesham, A. (1994). *International Journal of Theoretical Physics*, **33**, 1383.
Berman, M. S. (1992a). *International Journal of Theoretical Physics*, **31**, 1217.
Berman, M. S. (1992b). *International Journal of Theoretical Physics*, **31**, 1447.
Berman, M. S., and Som, M. M. (1990). *General Relativity and Gravitation*, **22**, 625.
Brans, C., and Dicke, R. H. (1961). *Physical Review*, **124**, 203.
Dirac, P. A. M. (1938). *Proceedings of the Royal Society of London A*, **165**, 199.
Hellings, R. (1983). GR-10 Report, Padova, Italy; *Physical Review Letters*, **51**, 1609.
Liddle, A. R., *et al.* (1992). *Physics Letters B*, **279**, 244.
Reasenberg, R. D., *et al.* (1979). *Astrophysical Journal*, **243**, L219.
Weinberg, S. (1972). *Gravitation and Cosmology*, Wiley, New York.
Whitrow, G., and Randall, D. (1951). *Monthly Notices of the Royal Astronomical Society*, **111**, 455.